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LETTER TO THE EDITOR

Pseudo-scaling in laboratory and in numerical simulations of turbulence

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Abstract. Stochastic systems with a stretched exponential form of probability density function (turbulence etc) are studied. A dimensionless moments is defined as $F_{np}(r) = f_p/f_n^{p/n}$, where $f_p(r)$ is the standard moment of order p ($p > n$). Pseudo-scaling (PS) is defined as the existence of power relationships $F_{np} \sim F_{nq}^{\rho}$, where exponent ρ_{npq} depends on p, q and n. The scaling is purely due to the way the given quantities are constructed and to the existence of the stretched exponential decay of the probability density functions. It is shown that for large enough n, p, q the pseudo-scaling takes place even if the ordinary scaling is broken. It is shown that there are two kinds of asymptotic pseudo-scaling: with $\rho_{npq} = (p-n)/(q-n)$ and with $\rho_{npq} = (p\ln(p/n))/(q\ln(q/n))$. If ordinary scaling also takes place in the system, then these two kinds of pseudo-scaling lead to two kinds of corresponding ordinary scaling laws. Agreement between the theoretical approach and experimental results of different authors is established for turbulent systems (laboratory and numerical simulations), both for situations where ordinary scaling takes place and for situations where ordinary scaling does not take place.

Scaling and corresponding power laws are widely used in the theory of stochastic systems. In the theory of turbulence, for instance, the scaling hypothesis is usually formulated in the terms of the moments of the space velocity differences

$$\langle \Delta u_r^p \rangle = \langle [u(x+r) - u(x)]^p \rangle \sim r^{\zeta_p} \tag{1}$$

where p is the order of the moment and ζ_p is the ordinary scaling exponent. In a neighbourhood of the ends of the scaling interval, the scaling law (1) should be broken. An analogous situation takes place in the cases when the scaling interval is not long enough. However, *dimensionless* moments (introduced for the first time in [1])

$$F_{np} = \frac{\langle \Delta u_r^p \rangle}{\langle \Delta u_r^n \rangle^{p/n}} \tag{2}$$

can fulfil the pseudo-scaling (PS) condition, which appears because of the existence of the power relationship

$$F_{np} \sim F_{nq}^{\rho} \tag{3}$$

even for situations when the ordinary scaling, (1), is broken (the idea of generalized self-similarity appeared for the first time in [2] where experimental evidence of this phenomenon are also given, see below).

It is known that probability density functions (PDFs) of *large* velocity differences in the turbulence have stretched exponential form

$$P(|\Delta u_r|) \sim \exp(-\lambda_r |\Delta u_r|^{m_r}) \tag{4}$$

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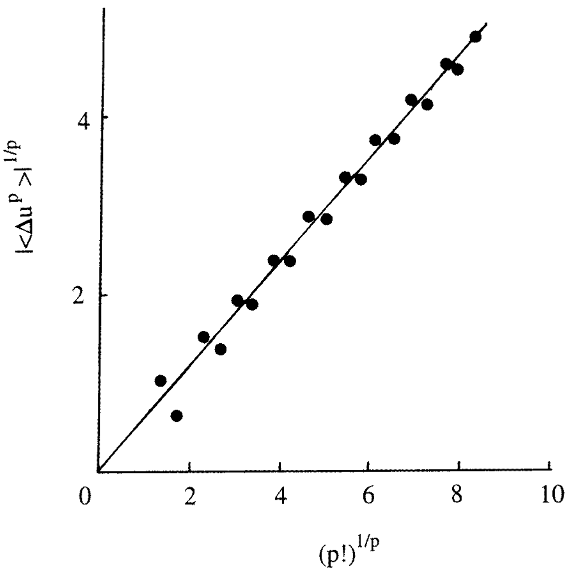

Figure 1. Moments (structure functions) of jet turbulence at space scale r corresponding to a neighbourhood of the molecular viscosity end of the scaling range: $|\langle \Delta u_r^p \rangle|^{1/p}$, as a function of the $(p!)^{1/p}$ (adapted from [5]).

where λ_r is some function on r (see, for instance, [3, 4]). In this section we will consider the simplest case with $m_r = 1$ (the general case will be considered later). Then one can estimate the *high-order* moments as

$$\langle |\Delta u_r|^p \rangle \sim \int_0^\infty |\Delta u|^p \exp(-\lambda_r |\Delta u|) \, \mathrm{d}\, |\Delta u|. \tag{5}$$

One cannot normalize the PDF due to the unknown form of the PDF for small values of $|\Delta u|$ which is significant for *small*-order moments. For large p, however, one can estimate

$$\langle |\Delta u_r|^p \rangle \simeq c_r \, p! \lambda_r^{-(p+1)} \tag{6}$$

where c_r is an unknown function of r (related to the normalization of the PDF). First of all, one can verify experimentally the dependence of the moments on p given by (6). Let us rewrite (6) in the form

$$\langle |\Delta u_r|^p \rangle^{1/p} \simeq (\chi_r)^{-1/p} \lambda_r^{-1} (p!)^{1/p} \tag{7}$$

where

$$\chi_r = \lambda_r / c_r. \tag{8}$$

Figure 1 shows recent experimental data obtained in a laboratory turbulent flow [5]. These data correspond to a neighbourhood of the short-wave end of the scaling range. Scales on axes in figure 1 are chosen so that a dependence like that of equation (7) corresponds (for large p) to a straight line intersecting the origin of the axes. One can see that there is agreement between representation (7) and the experimental data (dots in figure 1) beginning from $p = 4$. It should also be noted that if χ_r is very different from 1, then in figure 1 there should be a p-dependent deviation from the straight line behaviour.

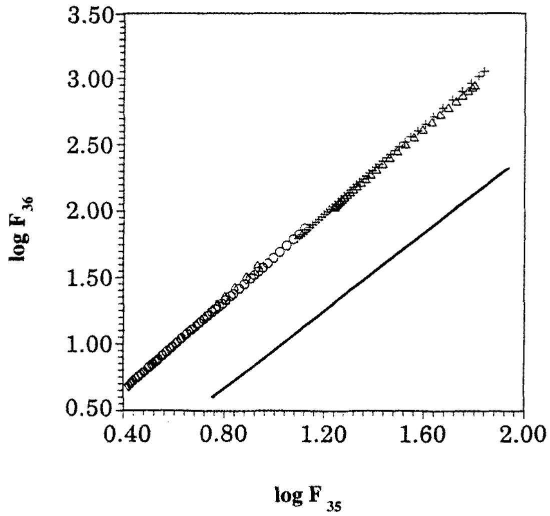

Figure 2. Log–log plot of F_{36} against F_{35} for different laboratory experiments and numerical simulations (adapted from [2]). The experimental data can be well fitted by relation (3) with $\rho_{365} \simeq 1.63$ given by (22) (whereas the continuous straight line corresponds to $\rho_{365} = 1.5$ given by (11)).

Using representation (6) (and (8)) one obtains

$$F_{np} \sim (\chi_r)^{\alpha_{np}} \tag{9}$$

where

$$\alpha_{np} = \frac{p}{n} - 1. \tag{10}$$

Since χ_r is a function of r one can use the parameter χ_r as a varied one that leads immediately to the pseudo-scaling law (3) even if the dependence of χ_r on r has no power form (i.e. with an absence of the ordinary self-similarity on the space scale r). Indeed, substituting (9) and (10) into (3) one obtains

$$\rho_{npq} = \frac{p-n}{q-n}. \tag{11}$$

This representation of ρ_{npq} is an asymptotic one (for large values of n, p, q). However, even for $n = 3$, equation (11) gives plausible values of exponent ρ_{npq}. Indeed, figure 2 (adapted from [2]) shows recent experimental data obtained in different laboratory turbulent flows (and for direct numerical simulations) in a space-scale range where molecular viscous effects are present and ordinary self-similarity is not observed. For these data, $n = 3$, $p = 6$, $q = 5$. The continuous straight line in figure 2 corresponds to $\rho_{365} = \frac{3}{2}$ obtained from (11) (see also the following where a better agreement with the experimental data will be obtained taking into account deviations of m_r from a constant value).

Let us consider the general case with m_r depending on r (cf earlier and [3, 4]). Then for large p

$$\langle|\Delta u_r|^p\rangle \sim c_r^* \lambda^{-(p+1)/m_r} \int_0^\infty x^{p/m_r} \mathrm{e}^{-x}\, \mathrm{d}x. \tag{12}$$

Since for large p

$$\int_0^\infty x^{p/m_r} \mathrm{e}^{-x}\, \mathrm{d}x \sim \left(\frac{p}{m_r e}\right)^{p/m_r} \tag{13}$$

one can estimate

$$\langle|\Delta u_r|^p\rangle \sim c_r^* \lambda^{-(p+1)/m_r} \left(\frac{p}{m_r e}\right)^{p/m_r}. \tag{14}$$

Then

$$F_{np} = \frac{\langle \Delta u_r^p \rangle}{\langle \Delta u_r^n \rangle^{p/n}} \sim (\chi_r^*)^{((p/n)-1)} \left(\frac{p}{n}\right)^{p/m_r} \tag{15}$$

where

$$\chi_r^* = \frac{\lambda_r^{1/m_r}}{c_r^*}. \tag{16}$$

One can rewrite (15) in a form similar to (9),

$$F_{np} \sim (\chi_r^*)^{\alpha_{np}^*(r)}$$

where

$$\alpha_{np}^*(r) = \left(\frac{p}{n} - 1\right) + \frac{p}{m_r} \log_{\chi_r^*}\left(\frac{p}{n}\right). \tag{17}$$

Then

$$\rho_{npq}^*(r) = \frac{((p/n) - 1) + (p/m_r)\log_{\chi_r^*}(p/n)}{((q/n) - 1) + (q/m_r)\log_{\chi_r^*}(q/n)}. \tag{18}$$

The exponent ρ_{npq}^* is independent of r (i.e. the pseudo-scaling exists) in two asymptotic cases: first

$$\left|\left(\frac{p}{n} - 1\right)\right| \gg \frac{p}{m_r}\left|\log_{\chi_r^*}\left(\frac{p}{n}\right)\right| \qquad \left|\left(\frac{q}{n} - 1\right)\right| \gg \frac{q}{m_r}\left|\log_{\chi_r^*}\left(\frac{q}{n}\right)\right| \tag{19}$$

and second

$$\frac{p}{m_r}\left|\log_{\chi_r^*}\left(\frac{p}{n}\right)\right| \gg \left|\left(\frac{p}{n} - 1\right)\right| \qquad \frac{q}{m_r}\left|\log_{\chi_r^*}\left(\frac{q}{n}\right)\right| \gg \left|\left(\frac{q}{n} - 1\right)\right|. \tag{20}$$

In the first case

$$\rho_{npq}^* \simeq \frac{p-n}{q-n} \tag{21}$$

i.e. in this case $\rho_{npq}^* \simeq \rho_{npq}$ (cf equation (11)). In the second case

$$\rho_{npq}^* \simeq \frac{p}{q}\frac{\ln(p/n)}{\ln(q/n)}. \tag{22}$$

It is interesting to note that for the experimental situations represented in figure 2 the experimental value of $\rho_{365} \simeq 1.65$ whereas estimation (22) gives $\rho_{365}^* \simeq 1.63$. This coincidence could be an indication that the asymptotical estimations of the PS-exponent are also applicable for moderate values of n, p, q.

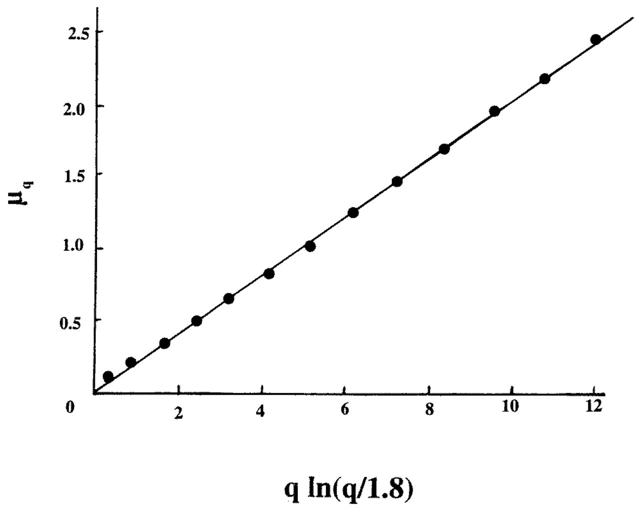

Figure 3. The scaling exponent, μ_q, of the turbulent energy dissipation field moments obtained in a recent turbulent experiment [8], against $q \ln(q/q_0)$.

In the cases where ordinary self-similarity takes place, the ordinary scaling asymptotics should be determined by the pseudo-scaling laws (if the PDFs of these systems have stretched exponential form). For the first type of pseudo-scaling this asymptotic can be found from the functional equation

$$\frac{\zeta_p - (p/n)\zeta_n}{\zeta_q - (q/n)\zeta_n} = \frac{p-n}{q-n} \tag{23}$$

which follows immediately from (1) and (11). It is easy to show that the general solution of (23) is a linear one,

$$\zeta_n = An + C \tag{24}$$

where A and C are some constants. This kind of linear scaling asymptotic is well known for turbulence (see, for example, [6, 7] and references therein).

The ordinary scaling asymptotic corresponding to the second kind of pseudo-scaling law can be found from the functional equation

$$\frac{\zeta_p - (p/n)\zeta_n}{\zeta_q - (q/n)\zeta_n} = \frac{p}{q}\frac{\ln(p/n)}{\ln(q/n)} \tag{25}$$

which follows immediately from (1) and (22). To obtain the general solution of this functional equation let us seek ζ_n in the following form,

$$\zeta_n = n\sigma(n) \tag{26}$$

where $\sigma(n)$ is an unknown function of the variable n. Then from (25) we obtain

$$\frac{\sigma_p - \sigma_n}{\sigma_q - \sigma_n} = \frac{\ln(p/n)}{\ln(q/n)}. \tag{27}$$

If we take derivatives of both parts of equation (27) with respect to the variable n we obtain

$$\frac{\mathrm{d}\sigma(n)}{\mathrm{d}n} = \frac{A}{n} \tag{28}$$

where $A = [\sigma_q - \sigma_p]/[\ln q - \ln p]$. Since the general solution of (28) is

$$\sigma(n) = A \ln(n/n_0) \tag{29}$$

(where n_0 is some constant) the corresponding general solution of (25) is

$$\zeta_n = An \ln(n/n_0). \tag{30}$$

To compare this result with a recent laboratory experiment [8], let us recall a relation between the scaling law of the velocity differences field (1) and the scaling law of the corresponding energy dissipation field ε_r [6, 8],

$$\langle \varepsilon_r^q \rangle \sim r^{-\mu_q}. \tag{31}$$

The scaling exponents from (1) and (31) are related by the following relationship [6, 8]

$$\zeta_p = \tfrac{1}{3} p - \mu_{p/3}. \tag{32}$$

Thus asymptotic (30) gives for μ_q an analogous representation

$$\mu_q = A^* q \ln(q/q_0) \tag{33}$$

where $A^* = -3A$ and q_0 is some constant.

Experimental evidence for the linear asymptotic of μ_q corresponding to (24) can be found in [9], whereas figure 3 shows recent experimental data (dots) obtained in a laboratory turbulent flow [8] and corresponding to (33). We choose the axes in figure 3 so that relation (33) is represented by a straight line intersecting the origin of the axes (it should also be noted that in the experiment [8] the ordinary scaling hypothesis only holds approximately). Thus, both pseudo-scaling laws (21) and (22) have corresponding ordinary scaling analogies (in the cases when the ordinary scaling takes place).

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